§3 Sequences

3.1 Sequences and Their Limits

Definition:

A sequence of real numbers (or sequence in $\mathbb{R}$) is defined as a function $X: \mathbb{N} \to \mathbb{R}$.

Usually, we write $x_n = X(n)$ and denote a sequence by $\{x_n\}$ or $\{x_n\}_{n=1}^{\infty}$

Examples:

1) $X: \mathbb{N} \to \mathbb{R}$ is defined by $X(n) = (\frac{1}{2})^n$

   Then $\{x_n\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$

2) (Fibonacci sequence)

   $\{f_n\}$ is given by $f_1 = f_2 = 1$ and $f_{n+2} = f_n + f_{n+1}$ for all $n \in \mathbb{N}$.

   $\therefore \{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$

"Definition": (Limit of a sequence)

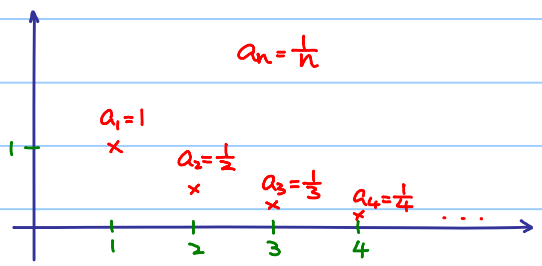
Suppose $\{x_n\}$ is a sequence of real numbers. $x \in \mathbb{R}$ is said to be a limit of the sequence $\{x_n\}$ if when n is getting larger and larger, x_n is getting closer and closer to x, we denote it by $\lim\limits_{n\to\infty} x_n = x$.

In this case, we also say $\{x_n\}$ converges to x. If the sequence has a limit, we say $\{x_n\}$ is a convergent sequence, otherwise it is a divergent sequence.

Caution: The limit x must be a real number and $+\infty, -\infty$ are NOT real numbers.

Examples:

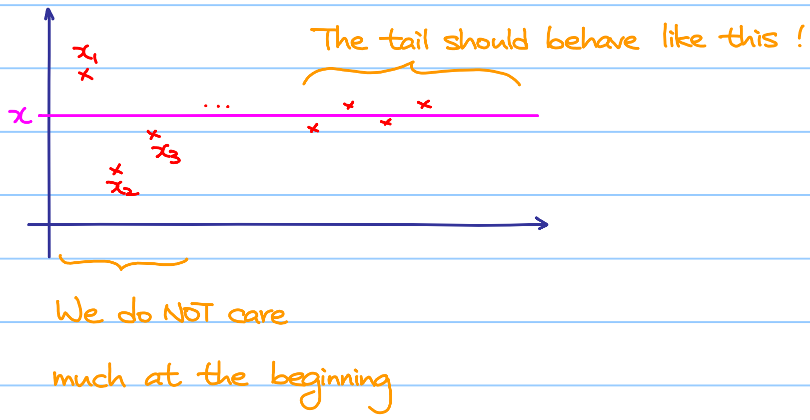
1) $x_n = \frac{1}{n}$



   Intutively, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

2) If $x_n = (-1)^n \frac{1}{n}$, intutively, $\lim\limits_{n\to\infty} x_n = 0$. (Remark: Try to represent $\{x_n\}$ graphically)

Naively, if $\lim_{n\to\infty} x_n = x$



How to formalize this idea ?

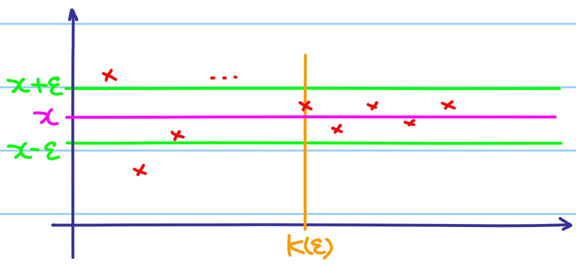
Definition : (Limit of a sequence)

A sequence $\{x_n\}$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$, if

For all $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ such that for all $n \geq K(\varepsilon)$, we have $|x_n - x| < \varepsilon$

$(\forall \varepsilon > 0)(\exists K(\varepsilon) > 0)(\forall n \geq K(\varepsilon))(|x_n - x| < \varepsilon)$

Geometrical meaning :



No matter how small $\varepsilon > 0$ we pick, if we draw a ε-tabular neighborhood of x, we can always find $K(\varepsilon) \in \mathbb{N}$ (depending on ε only) such that the terms $x_{K(\varepsilon)}, x_{K(\varepsilon)+1}, \cdots$ (terms at the tail) are all lying in the neighborhood.

Theorem: (Uniqueness of limit)

A sequence in $\mathbb{R}$ can have at most one limit.

proof:

Suppose $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$.

Claim: $a=b$.

By definition, let $\varepsilon>0$, there exist $K_1, K_2 \in \mathbb{N}$ such that

$|x_n - a| < \frac{\varepsilon}{2}$ for all $n \geq K_1$

$|x_n - b| < \frac{\varepsilon}{2}$ for all $n \geq K_2$

Consider $K = \max\{K_1, K_2\}$, if $n \geq K$

$$|x_n - a| + |x_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\geq |(x_n - a) - (x_n - b)| \quad \text{(Triangle inequality)}$$

$$= |a-b|$$

Since ε can be arbitrarily small, $a-b=0$ i.e. $a=b$.

Remark: Now, we can say *the* limit of $\{x_n\}$ instead of *a* limit of $\{x_n\}$.

Example:

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

What to do? Check the definition!

Idea: Let $\varepsilon > 0$,

State how to choose $K \in \mathbb{N}$ such that $|\frac{1}{n} - 0| < \varepsilon$ for all $n \geq K$.

$\Updownarrow$

$\frac{1}{n} \overset{?}{<} \varepsilon$

Think: if $n \geq K$, $\frac{1}{n} < \frac{1}{K} \overset{?}{<} \varepsilon$

Can we choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \varepsilon$? Yes, by (or corollary of) Archimedian property!

Here is how we write a proof:

Let $\varepsilon > 0$,

by Archimedian property, there exists $K \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < K$, i.e. $\frac{1}{K} < \varepsilon$.

Then for all $n \geq K$, we have $|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$.

$\therefore \lim_{n\to\infty} \frac{1}{n} = 0$.

Exercise :

By using the ε-definition, prove that

a) $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$

b) $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$

c) $\lim_{n\to\infty} \frac{2n^2}{n^2-2n+7} = 2$

Hint: $\left|\frac{2n^2}{n^2-2n+7} - 2\right| = \left|\frac{-(4n-14)}{n^2-2n+7}\right| = \left|\frac{4n-14}{n^2-2n+7}\right|$

Note : ① $4n-14 \geq 0 \iff n \geq \frac{7}{2}$ (so if $n \geq 4$, $4n-14 > 0$)

② $n^2-2n+7 = (n-1)^2+6 \geq 0$

③ $\frac{4n-14}{n^2-2n+7} = \underbrace{\frac{1}{n}} \cdot \frac{4-\frac{14}{n}}{1-\frac{2}{n}+\frac{7}{n^2}} \leq \frac{1}{n} \cdot \frac{4}{(\frac{1}{2})} = \frac{8}{n}$ ($\because 1-\frac{2}{n}+\frac{7}{n^2} \geq \frac{1}{2}$ when $n \geq 4$)

something we need in order to use Archimedian property !

④ $\frac{8}{n} \overset{?}{<} \varepsilon$

Here is how we write a proof :

Let $\varepsilon > 0$,

there exist $K' \in \mathbb{N}$ such that $\frac{8}{\varepsilon} < K'$ i.e. $\frac{8}{K'} < \varepsilon$ ④ (but we worry if $K' \geq 4$)

Take $K = \max\{K', 4\} \in \mathbb{N}$, then if $n \geq K$, we have

$$\left|\frac{2n^2}{n^2-2n+7} - 2\right| = \left|\frac{4n-14}{n^2-2n+7}\right|$$

$$= \frac{4n-14}{n^2-2n+7}$$ ($n \geq K \geq 4 \overset{①}{\Rightarrow} 4n-14 \geq 0$) ② $n^2-2n+7 \geq 0$

$$\leq \frac{1}{n} \cdot \frac{4-\frac{14}{n}}{1-\frac{2}{n}+\frac{7}{n^2}}$$

$$\leq \frac{1}{n} \cdot \frac{4}{(\frac{1}{2})}$$ ③ ($\because n \geq K \geq 4$)

$$= \frac{8}{n}$$

$$\leq \frac{8}{K}$$

$$< \varepsilon$$

$\therefore \lim_{n\to\infty} \frac{2n^2}{n^2-2n+7} = 2$

Exercise :

1) Try to write down the negation of the definition.

Ans : $x \in \mathbb{R}$ is not a limit of $\{x_n\}$ if

$(\exists \varepsilon > 0)(\forall K \in \mathbb{N})(\exists n \geq K(\varepsilon))(|x_n - x| \geq \varepsilon)$

Geometrical meaning ?

2) Prove that the sequence $x_n = \frac{1}{2}[1 + (-1)^n]$ is divergent.

Hint : Divide into 3 cases :

1) Why 0 cannot be the limit ?

2) Why 1 cannot be the limit ?

3) Why $x \neq 0, 1$ cannot be the limit ?

Definition :

Suppose $\{x_n\}$ is a sequence in $\mathbb{R}$ and $m \in \mathbb{N}$.

$\{x_{m+n}\}$ is another sequence in $\mathbb{R}$ that $\{x_{m+n}\} = \{x_{m+1}, x_{m+2}, \cdots\}$.

Exercise :

Prove that $\lim_{n\to\infty} x_{m+n} = x$ if and only if $\lim_{n\to\infty} x_n = x$.

Similarly : $\{x_{2n}\} = \{x_2, x_4, x_6, \cdots\}$

$\{x_{2n-1}\} = \{x_1, x_3, x_5, \cdots\}$

Exercise :

Prove that if $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n-1}$ exist.

Does the converse hold ?

Ans : The converse does not hold unless $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n-1}$.

3.2 Limit Theorem

Definition:

A sequence $\{x_n\}$ of real numbers is said to be **bounded** if there exists $M \geq 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

(Think: Geometrical meaning ?)

Theorem:

A convergent sequence of real numbers is bounded.

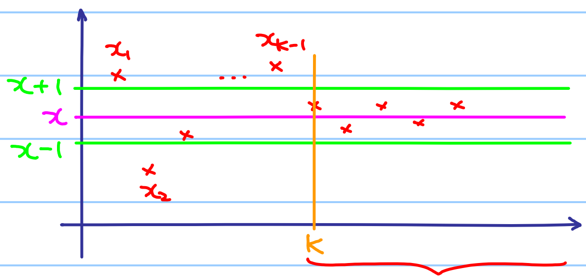
proof:

Suppose $\lim\limits_{n\to\infty} x_n = x$.

Take $\varepsilon = 1$, there exists $K \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq K$.

$\Rightarrow$ $|x_n| < |x| + 1$ for all $n \geq K$.

Take $M = \max\{|x_1|, |x_2|, \cdots, |x_{K-1}|, |x|+1\}$, the result follows!



controlled by $|x|+1$

Exercise:

Does the converse hold?

No! Consider $\{0, 1, 0, 1, \cdots\}$.

The converse holds if we further assume the monotonicity. (Discuss later!)

Definition:

A sequence $\{x_n\}$ of real numbers is said to be **bounded above (below)** if there exists $M \geq 0$ such that $x_n \leq M$ $(-M \leq x_n)$ for all $n \in \mathbb{N}$.

Theorem : (Algebraic properties)

Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

1) $\lim_{n\to\infty} x_n + y_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = x + y$

2) $\lim_{n\to\infty} x_n - y_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = x - y$

3) $\lim_{n\to\infty} x_n y_n = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n) = xy$

4) If $y \neq 0$, $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} = \frac{x}{y}$

proof :

1) Given $\varepsilon > 0$, there exist $K_1, K_2 \in \mathbb{N}$ such that

$|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq K_1$

$|y_n - y| < \frac{\varepsilon}{2}$ for all $n \geq K_2$

Take $K = \max\{K_1, K_2\}$, then for $n \geq K$, we have

$$
\begin{aligned}
&|(x_n + y_n) - (x + y)| \\
&= |(x_n - x) + (y_n - y)| \\
&\leq |x_n - x| + |y_n - y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

$\therefore \lim_{n\to\infty} x_n + y_n = x + y$.

2) Similar to (1) , leave as an exercise.

3) Hint : $|x_n y_n - xy|$

$$
\begin{aligned}
&= |x_n y_n - x y_n + x y_n - xy| \\
&\leq \underbrace{|y_n|}_{\text{bounded}} |x_n - x| + |x||y_n - y|
\end{aligned}
$$

4) Hint : $\left|\frac{x_n}{y_n} - \frac{x}{y}\right|$

$$
\begin{aligned}
&= \frac{1}{|y_n y|} |x_n y - x y_n| \\
&= \frac{1}{|y_n y|} |x_n y - x_n y_n + x_n y_n - x y_n| \\
&\leq \underbrace{\frac{1}{|y_n y|}}_{\text{bounded}} \left(\underbrace{|x_n|}_{\text{bounded}} |y_n - y| + \underbrace{|y_n|}_{\text{bounded}} |x_n - x| \right)
\end{aligned}
$$

Exercises :

1) Prove that if $\lim_{n\to\infty} x_n = x$ and $c \in \mathbb{R}$, then $\lim_{n\to\infty} cx_n = cx$.

(Prove by checking the ε-definition or applying (3) with $y_n = c$)

2) Fill in the details of the previous theorem.

3) Prove that $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$.

(Hint: Nothing, but $|x_n - 0| = ||x_n| - 0|$)

Theorem :

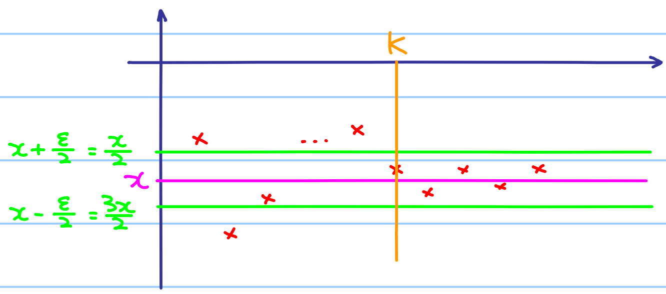
If $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $x \geq 0$.

proof :

Suppose the contrary, $x < 0$.

Take $\varepsilon = \frac{|x|}{2} = -\frac{x}{2}$, there exists $K \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq K$.

In particular, $|x_K - x| < \varepsilon \Rightarrow x_K < x + \varepsilon = \frac{x}{2} < 0$ (Contradiction !)



Remarks :

1) Any weaker version of the above theorem ?

Yes, $x_n \geq 0$ for sufficiently large n ! (Anyway, what really matters is the tail !)

2) If $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, does it imply $x > 0$?

No ! Consider $x_n = \frac{1}{n} > 0$.

Theorem :

If $x_n \geq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $x \geq y$.

proof :

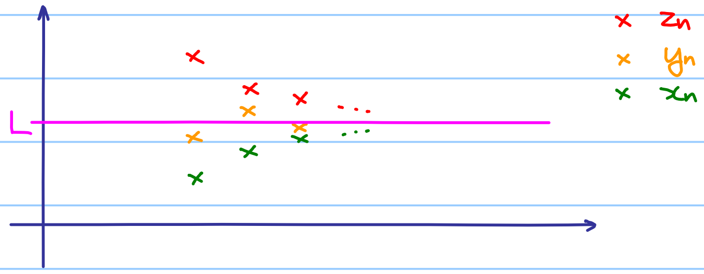
Apply the previous theorem to $z_n = x_n - y_n \geq 0$.

Exercise :

Prove that if $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $a \leq x \leq b$.

Theorem : (Sandwich theorem)

Suppose $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$. Then $\{y_n\}$ is convergent and $\lim_{n\to\infty} y_n = L$.



proof :

Given $\varepsilon > 0$, there exist $K_1, K_2 \in \mathbb{N}$ such that

$|x_n - L| < \varepsilon$ for all $n \ge K_1$ $\Rightarrow$ $L - \varepsilon < x_n$ for all $n \ge K_1$

$|z_n - L| < \varepsilon$ for all $n \ge K_2$ $\Rightarrow$ $z_n < L + \varepsilon$ for all $n \ge K_2$

Take $K = \max\{K_1, K_2\}$, then for $n \ge K$, we have

$L - \varepsilon < x_n \le y_n \le z_n < L + \varepsilon$

$\Rightarrow |y_n - L| < \varepsilon$

$\therefore \lim_{n\to\infty} y_n = L$

Example :

Find $\lim_{n\to\infty} \frac{1}{n}\sin n$ (Note : $\lim_{n\to\infty} \sin n$ does (may) NOT exist .)

$-1 \le \sin n \le 1$

$-\frac{1}{n} \le \frac{1}{n}\sin n \le \frac{1}{n}$

and $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

By sandwich theorem, $\lim_{n\to\infty} \frac{1}{n}\sin n = 0$

Main issues in this example : ① $|\sin n| \le 1$ for all $n \in \mathbb{N}$ (bounded)

② $\lim_{n\to\infty} \frac{1}{n} = 0$

We can generalize this result as a theorem.

Theorem:

Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bounded, then $\lim_{n\to\infty} x_n y_n = 0$.

proof:

By assumption that $\{y_n\}$ is bounded,

there exists $M \geq 0$ such that $|y_n| \leq M$ (i.e. $-M \leq y_n \leq M$) for all $n \in \mathbb{N}$

Note: $-|x_n| \leq x_n \leq |x_n|$ for all $n \in \mathbb{N}$

$\therefore -M|x_n| \leq x_n y_n \leq M|x_n|$ for all $n \in \mathbb{N}$

$\lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} |x_n| = 0$

$\Rightarrow \lim_{n\to\infty} -M|x_n| = \lim_{n\to\infty} M|x_n| = 0$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty} x_n y_n = 0$.

Exercises:

1) Show that $\lim_{n\to\infty} \frac{(-1)^n}{n} \sin n = 0$

2) We know that $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$.

Is it true that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} |x_n| = |x|$ for $x \neq 0$?

Ans: "$\Rightarrow$" is true (try to prove) but "$\Leftarrow$" is false!

3.3 Monotone Sequences

Recall: $\{x_n\}$ is converge $\underset{\not\Leftarrow}{\Rightarrow}$ $\{x_n\}$ is bounded.

However, can we add suitable assumption so that "$\Leftarrow$" is true?

Definition:

Let $\{x_n\}$ be a sequence of real numbers.

$\{x_n\}$ is said to be increasing (decreasing) if $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$) for all $n \in \mathbb{N}$.

$\{x_n\}$ is said to be a monotone if it is either increasing or decreasing.

Example:

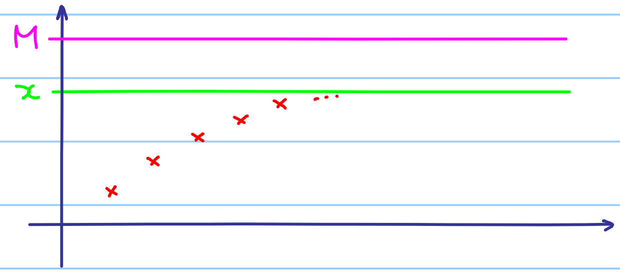
If $x_n = 2^n$ for all $n \in \mathbb{N}$, then clearly, $\{x_n\}$ is increasing.

However, $\{x_n\}$ is divergent.

Theorem: (Monotone Convergence Theorem)

If $\{x_n\}$ is a sequence of real numbers such that it is bounded above (below) and increasing, then $\{x_n\}$ is convergent.

Idea:



Caution: Even $x_n \le M$ for all $n \in \mathbb{N}$, it does NOT imply $\lim_{n\to\infty} x_n = M$!

Why not M ? M may NOT be the tightest one !

The tightest one $= x = \sup\{x_n : n \in \mathbb{N}\}$

Claim: $\lim_{n\to\infty} x_n = x$.

proof:

Given $\varepsilon > 0$,

$x - \varepsilon$ is NOT an upper bound of $\{x_n : n \in \mathbb{N}\}$

$\Rightarrow$ there exists $K \in \mathbb{N}$ such that $x - \varepsilon < x_K < x$.

Then for $n \ge K$, we have $x - \varepsilon < x_K \le x_n \le x < x + \varepsilon$

$\{x_n\}$ is increasing.

$\Rightarrow |x_n - x| < \varepsilon$

$\therefore \lim_{n\to\infty} x_n = x$.

Theorem: (Euler's Number e)

Let $\{e_n\}$ be a sequence of real numbers defined by $e_n = (1+\frac{1}{n})^n$ for $n \in \mathbb{N}$.

Then $\{e_n\}$ is convergent and we denote $e := \lim_{n\to\infty} e_n$.

proof:

1) Claim: $\{e_n\}$ is increasing.

$$\begin{aligned} e_n = (1+\tfrac{1}{n})^n &= \sum_{r=0}^{n} C^n_r \cdot (\tfrac{1}{n})^r \\ &= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \cdot (\tfrac{1}{n})^r \\ &= \sum_{r=0}^{n} \frac{1}{r!} \cdot [n(n-1)(n-2)\cdots(n-r)] \cdot (\tfrac{1}{n})^r \\ &= \sum_{r=0}^{n} \frac{1}{r!} \cdot 1 \cdot (1-\tfrac{1}{n}) \cdot (1-\tfrac{2}{n}) \cdots (1-\tfrac{r-1}{n}) \end{aligned}$$

Similarly, $e_{n+1} = (1+\frac{1}{n+1})^{n+1} = \sum_{r=0}^{n+1} \frac{1}{r!} \cdot 1 \cdot (1-\frac{1}{n+1})(1-\frac{2}{n+1}) \cdots (1-\frac{r-1}{n+1})$

$\therefore e_n \le e_{n+1}$ for all $n \in \mathbb{N}$.

2) Claim: $\{e_n\}$ is bounded above by 3.

$$\begin{aligned} e_n &= \sum_{r=0}^{n} \frac{1}{r!} \cdot 1 \cdot \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) \\ &\leq \sum_{r=0}^{n} \frac{1}{r!} \\ &\leq 1+\sum_{r=1}^{n}\frac{1}{2^{r-1}} \qquad \text{Note: } \frac{1}{r!} \leq \frac{1}{2^{r-1}} \text{ if } r \geq 1 \quad \text{(Why?)} \\ &\leq 1+\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right) \\ &< 3 \end{aligned}$$

Combining (1) and (2), the result follows by monotone convergence theorem.

3.4 Subsequences and the Bolzano-Weierstrass Theorem

Definition:

Let $\{x_n\}$ be a sequence of real numbers and $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then $\{x_{n_k}\}$ given by $\{x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots\}$ is called a **subsequence** of $\{x_n\}$.

Natural result we expect:

Theorem:

If $\{x_n\}$ is a sequence of real numbers that converges to x, then any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x as well, we write $\lim_{n_k\to\infty} x_{n_k} = x$ or $\lim_{k\to\infty} x_{n_k} = x$.

proof: (Exercise)

Not surprising, but the contrapositive is actually more useful.

Theorem: (Divergence Criteria)

a) There exists a divergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

b) There exist subsequences $\{x_{n_k}\}$ and $\{x'_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} \neq \lim_{k\to\infty} x'_{n_k}$.

$\{x_n\}$ is divergent if either (a) or (b) holds.

Exercise:

Prove that the following sequences diverge.

a) $\{x_n\} = \{0,1,0,2,0,3,\cdots\}$ b) $\{x_n\} = \{-1,1,-1,1,-1,1,\cdots\}$ c) $x_n = \sin\frac{n\pi}{2}$

d) $x_n = \sin n$ (Hint: prove there exist $\{x_{n_k}\}$ and $\{x'_{n_k}\}$ such that $x_{n_k} \geq \frac{1}{\sqrt{2}}$ and $x'_{n_k} \leq \frac{1}{\sqrt{2}}$.)

While NOT every sequence is monotone,

it is surprising that every sequence has a monotone subsequence.

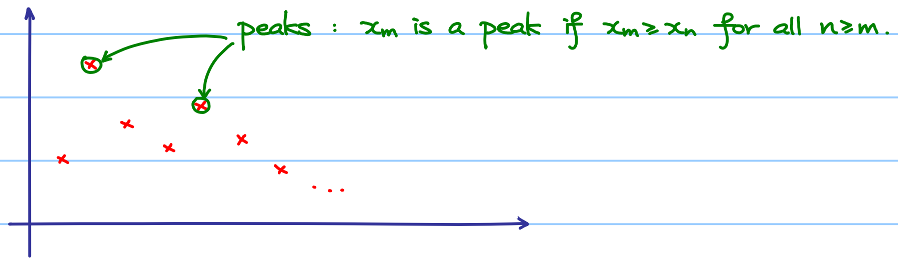
Theorem: (Monotone subsequence theorem)

If $\{x_n\}$ is a sequence of real numbers,

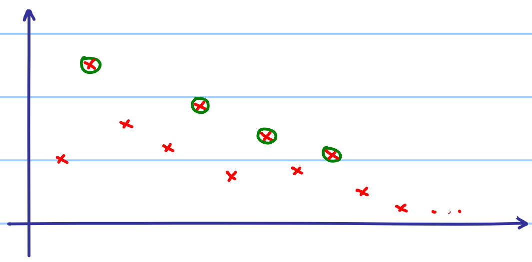
then there exists a monotone subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

proof:

Idea:

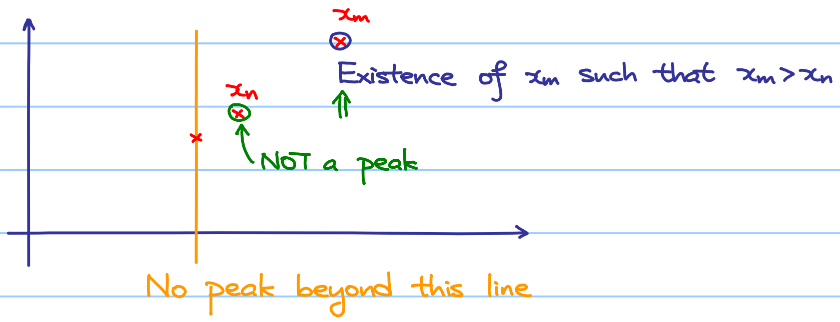


Case 1: $\{x_n\}$ has infinitely many peaks.



Those peaks form a decreasing sequence.

Case 2: $\{x_n\}$ has finite peaks.



Repeating this argument to get an increasing subsequence.

Theorem: (Bolzano-Weierstrass theorem)

A bounded sequence of real numbers has a convergent subsequence.

proof:

Monotone subsequence theorem + Monotone convergence theorem

Examples:

1) $x_n = n$ for all $n \in \mathbb{N}$

$\{x_n\}$ has no convergent subsequence! ($\{x_n\}$ is unbounded)

2) $x_n = \sin n$ for all $n \in \mathbb{N}$.

$\{x_n\}$ is bounded and divergent. However, Bolzano-Weierstrass theorem guarantees the existence of a convergent subsequence. Unfortunately, it is hard to write it down explicitly.

Come back to the question:

$$\lim_{n\to\infty} x_n = x \underset{\not\Leftarrow}{\Rightarrow} \lim_{k\to\infty} x_{n_k} = x.$$

"$\Leftarrow$" may NOT true because $\{x_{n_k}\}$ may NOT "contain the whole tail" of $\{x_n\}$.

(Only looking at one subsequence!)

However, how about looking at ALL subsequences?

Theorem:

Let $\{x_n\}$ be a sequence of real numbers.

$\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converges to x.

proof:

"$\Rightarrow$" Discussed before.

"$\Leftarrow$" Suppose the contrary,

there exists $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}$, there exists $n \geq k$ such that $|x_n - x| \geq \varepsilon_0$ —(*)

Let $k=1$, apply (*), we have $n_1 \geq k = 1$ such that $|x_{n_1} - x| \geq \varepsilon_0$

Let $k = n_1$, apply (*), we have $n_2 \geq k = n_1$ such that $|x_{n_2} - x| \geq \varepsilon_0$

$\vdots$

Repeating (*) to obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k} - x| \geq \varepsilon_0$ —(**)

Note: $\{x_n\}$ is bounded $\Rightarrow$ $\{x_{n_k}\}$ is bounded

By Bolzano-Weierstrass theorem $\{x_{n_k}\}$ has a convergent subsequence,

but the limit of it cannot be x because of (**).

It contradicts to the assumption that every subsequence of $\{x_n\}$ converges to x.

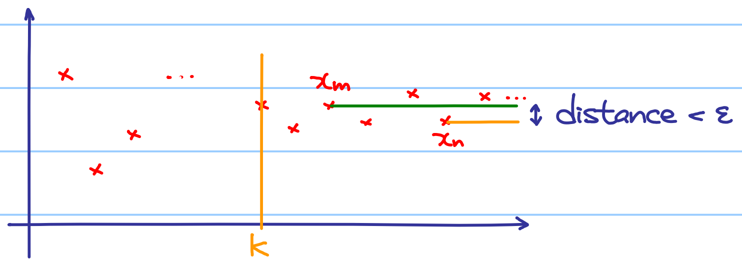
3.5 Cauchy Criterion

Definition:

A sequence $\{x_n\}$ of real numbers is said to be a **Cauchy sequence** if for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $m, n \geq K$, we have $|x_m - x_n| < \varepsilon$.

$(\forall \varepsilon > 0)(\exists K \in \mathbb{N})(\forall m, n \geq K)(|x_m - x_n| < \varepsilon)$

Geometrical meaning:



No matter how small $\varepsilon > 0$ we pick,
we can always find $K \in \mathbb{N}$ such that for any two points beyond the line,
the distance between them is less than ε.

Exercise:

Write down the negation of the definition.

Ans: $\{x_n\}$ is NOT a Cauchy sequence if $(\exists \varepsilon > 0)(\forall K \in \mathbb{N})(\exists m, n \geq K)(|x_m - x_n| \geq \varepsilon)$

Natural Question:

$\{x_n\}$ is convergent $\overset{?}{\Leftrightarrow}$ $\{x_n\}$ is Cauchy   Ans: Yes!

Theorem:

Let $\{x_n\}$ be a sequence of real numbers.
$\{x_n\}$ is a convergent sequence if and only if $\{x_n\}$ is a Cauchy sequence.

proof:

"$\Rightarrow$" Suppose $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for any $n \geq K$, we have $|x_n - x| < \frac{\varepsilon}{2}$.

Consider $m, n \geq K$, we have $|x_m - x| < \frac{\varepsilon}{2}$ and $|x_n - x| < \frac{\varepsilon}{2}$.

Then $|x_m - x_n| = |x_m - x + x - x_n|$

$$\leq |x_m - x| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

"$\Leftarrow$" Suppose $\{x_n\}$ is a Cauchy sequence.

Step 1: Prove that $\{x_n\}$ is bounded.

Step 2: By Bolzano-Weierstrass theorem $\{x_{n_k}\}$ has a convergent subsequence, and let $\lim_{k\to\infty} x_{n_k} = x$.

Step 3: Show that $\{x_n\}$ converges to x.

proof of (1): Take $\varepsilon = 1$, there exists $k \in \mathbb{N}$ such that $|x_n - x_k| < 1$

Take $M = \{|x_1|, |x_2|, \cdots, |x_{k-1}|, |x_k|+1\}$, then $|x_n| \le M$ for all $n \in \mathbb{N}$.

proof of (3): Given $\varepsilon > 0$,

there exists $K \in \mathbb{N}$ such that for any $m, n \ge K$,

we have $|x_m - x_n| < \frac{\varepsilon}{2}$.

Since $\{x_{n_k}\}$ converges to x,

there exists $n_j \ge K$ such that $|x_{n_j} - x| < \frac{\varepsilon}{2}$.

Then for all $n \ge K$,

$$
\begin{aligned}
|x_n - x| &= |x_n - x_{n_j} + x_{n_j} - x| \qquad \text{(Use } x_{n_j} \text{ as a "bridge")} \\
&\le |x_n - x_{n_j}| + |x_{n_j} - x| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

Remark:

Theorem:

Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

$\{x_n\}$ converges to a point in $\mathbb{R}$ if and only if $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$.

Question: Can we replace $\mathbb{R}$ by other space X?

Answer: No! Consider $X = (0,1)$ or $X = \mathbb{Q}$, the theorem does NOT hold anymore!

(e.g. $x_n = \frac{1}{n} \in (0,1)$ for all $n \in \mathbb{N}$, $\{x_n\}$ is a Cauchy sequence in $(0,1)$,

however $\lim_{n\to\infty} \frac{1}{n} = 0$ which is NOT a point in $(0,1)$!)

It is a main issue studied in Functional Ananlysis.

Exercises:

1) Prove directly that the following sequences are Cauchy sequences.

(Instead of proving they converge and apply the above theorem)

a) $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$

b) $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

2) Prove directly that $x_n = 1 + (-1)^n$ is NOT a Cauchy sequence.

3.6 Proper Divergent Sequences

Note: The limit of a sequence must be a real number, but ∞ and $-\infty$ are just conventions, so we cannot say "$\{x_n\}$ converges to ∞ or $-\infty$".

However, it is still worth to define clearly what $\lim_{n\to\infty} x_n = \infty$ / $-\infty$ means.

Definition: (Divergent but NOT too bad)

We say $\lim_{n\to\infty} x_n = \infty$ $(-\infty)$ if for all $M \in \mathbb{R}$, there exists $K \in \mathbb{N}$ such that $x_n \geq M$ $(x_n \leq M)$ for all $n \geq K$.

We say $\{x_n\}$ is properly divergent in case we have $\lim_{n\to\infty} x_n = \infty$ or $\lim_{n\to\infty} x_n = -\infty$.

Examples:

1) $\lim_{n\to\infty} n = \infty$

By Archimedian property, for all $M \in \mathbb{R}$, there exists $K \in \mathbb{N}$ such that $K \geq M$.

Then for all $n \geq K$, we have $n \geq K \geq M$.

2) $\lim_{n\to\infty} n^2 = \infty$

(How to prove?)

Generalizing (1):

Theorem:

If $\{x_n\}$ is an unbounded increasing (decreasing) sequence, then $\lim_{n\to\infty} x_n = \infty$ $(-\infty)$.

proof: (Exercise)

Theorem :

Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$ such that $x_n \le y_n$ for all $n \in \mathbb{N}$.

- $\lim\limits_{n\to\infty} x_n = \infty \Rightarrow \lim\limits_{n\to\infty} y_n = \infty$
- $\lim\limits_{n\to\infty} y_n = -\infty \Rightarrow \lim\limits_{n\to\infty} x_n = -\infty$

proof : (Exercise)

Remark :

It is particularly useful in studying infinite series.

Exercise :

Prove that $\lim\limits_{n\to\infty} \frac{n^2+1}{n} = \infty$